UNIT 6

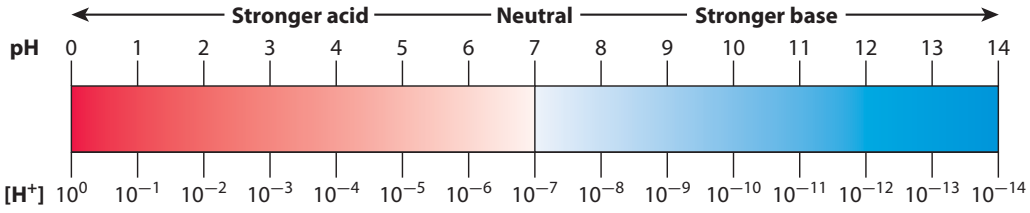
Exponential and Logarithmic Functions and Equations

Mathematics and Archaeology

Archaeologists study human history and activity through the recovery and analysis of artifacts and other physical remains. Did you know that it's also possible to determine if there was a human presence at a particular site even if no artifacts are found? Methods discovered in the late twentieth century that combine chemistry, geology, and topography have made this possible. Now, archaeological digs where no artifacts are found can still provide important information about the past.

While the presence of artifacts is fairly conclusive proof of a cultural presence, the lack of artifacts doesn't necessarily disprove a cultural presence. In fact, the multidisciplinary methods mentioned above can provide evidence of a cultural presence even without artifacts. By performing a chemical analysis on soil samples from a site where artifacts are found, the impact of the culture on the soil can be determined. Artifacts such as mollusk shells and animal bones break down and raise the concentration of calcium in the soil and affect the pH of the soil, making it more alkaline than normal. Even if digs in other locations do not yield cultural artifacts, if the soil pH in both sites is similar, a link can be established between the cultures of the peoples in the two sites.

A measure of the pH level of soil is really a measure of the concentration of hydrogen ions in the soil (pH stands for "potential of hydrogen"). The pH scale runs from 0 to 14, where 7 is neutral. Alkalinity increases as the numbers increase, and acidity increases as the numbers decrease.



Since each whole-number increase in pH represents a ten-fold decrease in the concentration of hydrogen atoms, the pH scale is an example of a logarithmic scale. Specifically,

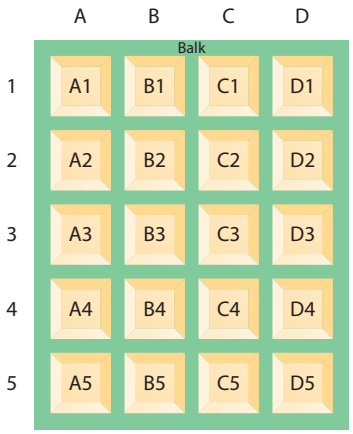
$$pH = -\log[H^+].$$

Furthermore, since pH is defined in terms of the common logarithm, then the hydrogen ion concentration can be found from the pH, using the inverse, exponential function,

$$[H^+] = 10^{-PH}.$$

Exercises

Archaeologists often make excavation grids like the one below on sites where they are searching for artifacts. An excavation grid consists of squares of earth separated by partitions called balks. Suppose a group is excavating a section of a large field. The group creates an excavation grid such that the area of each grid square and its surrounding balks is 25 square meters. Each person can excavate a grid square in 20 days.



1. **a.** How many total square meters can a person excavate in 1 day? in 2 days? in 3 days?

b. Is the total number of square meters excavated by a person after n days an arithmetic or a geometric sequence? Write a recursive rule and an explicit rule for a sequence that models the situation.

c. If there are 10 people in the group, how many total square meters will the group have excavated in 10 days?

Archaeologists use the pH scale to rate the acidity of soils. The pH of a soil is given by the equation

$$pH = -\log[H^+],$$

where $[H^+]$ is the soil's hydrogen ion concentration in moles per liter. (An ion is an electrically charged atom, and a mole of ions contains about 6.02×10^{23} ions.) The greater a soil's pH, the more alkaline (or less acidic) the soil is.

2. **a.** At an excavation site, archaeologists found that the soil could be divided into three distinct layers. The hydrogen ion concentrations of the uppermost, middle, and bottommost layers of soil were 4.0×10^{-7}, 1.6×10^{-7} and 2.0×10^{-6}, respectively, all in units of moles per liter. Find the pH of each soil layer, rounded to the tenths place.

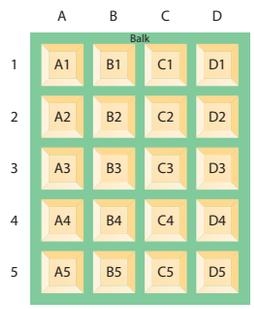
b. The presence of human artifacts increases a soil's alkalinity. Which of the soil layers in part (a) do you think contained the most artifacts? Explain.

3. At a separate excavation site, the pH of one soil layer was found to be 6.0, while that of another layer was found to be 6.6. Find the hydrogen ion concentration, $[H^+]$, in moles per liter, in each of the two layers. Express your answer in scientific notation using two significant digits.

4. Suppose the difference in pH of two soils, A and B, is 2. Compare the hydrogen ion concentrations in the soils.

Exercises

Archaeologists often make excavation grids like the one below on sites where they are searching for artifacts. An excavation grid consists of squares of earth separated by partitions called balks. Suppose a group is excavating a section of a large field. The group creates an excavation grid such that the area of each grid square and its surrounding balks is 25 square meters. Each person can excavate a grid square in 20 days.



1. **a.** How many total square meters can a person excavate in 1 day? in 2 days? in 3 days?
1.25 m²; 2.5 m²; 3.75 m²

b. Is the total number of square meters excavated by a person after n days an arithmetic or a geometric sequence? Write a recursive rule and an explicit rule for a sequence that models the situation.
arithmetic; $f(1) = 1.25$ and $f(n - 1) = f(n) + 1.25$; $f(n) = 1.25n$

c. If there are 10 people in the group, how many total square meters will the group have excavated in 10 days? **125 square meters**

Archaeologists use the pH scale to rate the acidity of soils. The pH of a soil is given by the equation

$$\text{pH} = -\log[\text{H}^+],$$

where $[\text{H}^+]$ is the soil's hydrogen ion concentration in moles per liter. (An ion is an electrically charged atom, and a mole of ions contains about 6.02×10^{23} ions.) The greater a soil's pH, the more alkaline (or less acidic) the soil is.

2. **a.** At an excavation site, archaeologists found that the soil could be divided into three distinct layers. The hydrogen ion concentrations of the uppermost, middle, and bottommost layers of soil were 4.0×10^{-7}, 1.6×10^{-7} and 2.0×10^{-6}, respectively, all in units of moles per liter. Find the pH of each soil layer, rounded to the tenths place. **6.4; 6.8; 5.7**

b. The presence of human artifacts increases a soil's alkalinity. Which of the soil layers in part (a) do you think contained the most artifacts? Explain. **See below.**

3. At a separate excavation site, the pH of one soil layer was found to be 6.0, while that of another layer was found to be 6.6. Find the hydrogen ion concentration, $[\text{H}^+]$, in moles per liter, in each of the two layers. Express your answer in scientific notation using two significant digits. **1.0×10^{-6}; 2.5×10^{-7}**

4. Suppose the difference in pH of two soils, A and B, is 2. Compare the hydrogen ion concentrations in the soils.
Sample answer: The concentration in one soil is 100 times that of the other.

2.b. Sample answer: The middle layer, because it has the highest pH and is therefore the most alkaline.