Problem Solving in Math Expressions: Supporting the CCSS Operations and Algebraic Thinking Learning Path

Building a New Standard of Success

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OVERVIEW

Math Expressions author, Dr. Karen Fuson, taught math for three years in Chicago. Her experiences there suggested features of the Children's Math Worlds (CMW) Research Project, her ten-year study of how to effectively teach students math from an early age. The project focused on ways that students around the world understand mathematical concepts, approach problem solving, and learn to do computation.

The CMW Research Project examined how teachers can build conceptual supports, including special language, drawings, manipulatives, and classroom communication methods, to facilitate mathematical competence. The results from these ten plus years of research, and from the research of others, helped create the problem-solving approaches that are used in *Math Expressions*.

RESEARCHED-BASED PROBLEM-SOLVING STRATEGIES

A major focus of the National Science Foundation-funded research that underlies *Math Expressions* was identifying ways to help students understand and be able to explain word problems and computational methods, while also bringing students to accuracy and fluency in these areas. To encourage students to become reflective and resourceful problem solvers, *Math Expressions* uses an algebraic approach to problem solving in which students solve an ambitious range of word problems by representing the situation, and then solving using that representation. Students must understand, represent, and solve the situation, and then check for reasonableness. Students make math drawings, use researched-based accessible numerical drawings, and engage in Math Talk. These help students organize information in word problems and find solutions. These approaches make a variety of word problems accessible to all students, including struggling students and English language learners.

THE CCSS OA DOMAIN

The OA: Operations and Algebraic Thinking domain drew on the same international research used in developing *Math Expressions*. These OA standards are deeply consistent with the content and approaches already in *Math Expressions*.

Figure 1. Add To/Take From, Put Together/Take Apart, and Comparison Problems

Problem Type	Situation Equation	Word Problem
In Add To/Take From situations, a quantity is given, it is then modified by a change—some quantity is either added to or taken from—and a new quantity results.		
Add To		
Result Unknown	$9 + 4 = \square$	Dan had 9 cherries. Then he picked 4 more. How many does he have now?
Change Unknown	$9 + \square = 13$	Dan had 9 cherries. Then he picked some more. Now he has 13 cherries. How many did he pick?
Start Unknown	$\square + 4 = 13$	Dan had some cherries. Then he picked 4 more. He has 13 cherries. How many did he start with?
Take From		
Result Unknown	$13 - 9 = \square$	Dan had 13 cherries. Then he ate 9 of them. How many does he have now?
Change Unknown	$13 - \square = 4$	Dan had 13 cherries. Then he ate some of them. Now he has 4 cherries. How many did he eat?
Start Unknown	$\square - 9 = 4$	Dan had some cherries. Then he ate 9 of them. Now he has 4 cherries. How many did he start with?
In Put Together/Take Apart situations, all objects are present from the start, and nothing is added or taken away.		
Total Unknown		
Put Together	$9 + 4 = \square$	Ana put 9 dimes and 4 nickels in her pocket. How many coins did she put in her pocket?
Take Apart	$\square = 9 + 4$	Ana put 9 coins in her purse and 4 coins in her bank. How many coins did she have in the beginning?
Addend (Partner) Unknown		
Put Together	$13 = 9 + \square$ or $9 + \square = 13$	Ana put 13 coins in her pocket. 9 are dimes and the rest are nickels. How many nickels are in her pocket?
Take Apart	$13 = 9 + \square$	Ana had 13 coins. She put 9 in her purse and the rest in her bank. How many coins did she put in her bank?
In Additive Comparison situations, the problems involve finding or knowing how many more or less one quantity is than another quantity. The comparison can be said in two ways.		
Difference Unknown	$9 + \square = 13$ or $13 - \square = 9$	Ali has 9 balloons. Lisa has 13 balloons. How many more balloons does Lisa have than Ali? Ali has 9 balloons. Lisa has 13 balloons. How many fewer balloons does Ali have than Lisa?
Bigger Amount Unknown	$9 + 4 = \square$	Ali has 9 balloons. Lisa has 4 more than Ali. How many balloons does Lisa have? Ali has 9 balloons. He has 4 fewer than Lisa. How many balloons does Lisa have?

Figure 2. Equal Groups, Array/Area, and Comparison Problems

Problem Type	Word Problem
In Equal Groups situations, the problem involves objects that are separated into groups with the same number in each group.	
Multiplication Product Unknown	Amy has 5 cousins. She is making 2 puppets for each cousin. How many puppets will Amy need to make?
Division Number of Groups Unknown	Amy made 10 puppets for her cousins. Each cousin will get 2 puppets. How many cousins does Amy have?
Division Group Size Unknown	Amy made 10 puppets to divide equally among her 5 cousins. How many puppets will each cousin get?
In Array situations, the problems involve objects that are organized in equal rows and columns that are not connected.	
Multiplication Product Unknown	A garden has 5 rows and 2 columns of bean plants. How many plants are there in all?
Division Column (or Row) Unknown	A garden has 10 bean plants in 5 equal rows. How many columns does it have?
In rectangular Area situations, the square units form an array of squares pushed together to make a rectangle.	
Multiplication Product Unknown	The garden is 5 yards wide and 2 yards long. What is its area?
Division Length (or Width) Unknown	The garden is 10 square yards in area. It is 5 yards wide. How long is it?
In Multiplicative Comparison situations, the problems involve one quantity that is a number of times as many as another quantity.	
Multiplication Larger Unknown	Bill has 2 apples. Kim has 5 times as many apples as Bill. How many apples does Kim have?
Division Smaller Unknown	Kim has 10 apples. Bill has Kim's apples divided by 5. How many apples does Bill have?
Division Difference Unknown	Kim has 10 apples. Bill has 2 apples. Kim has how many times as many apples as Bill has?

Note: In the Array and Area situations, the factors play similar roles, so only one example of an unknown factor (division) is given above. But the other factor can be the unknown (e.g., the row, widths, and the other combined quantity). The situation equation for all multiplication situations is 2 x 5 = □. Division situations can be expressed as divisions, 10 ÷ 5 = □, or as an unknown factor, □ x 5 =10.

UNDERSTAND THE SITUATION

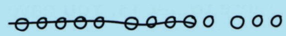
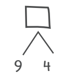
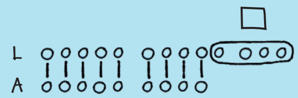
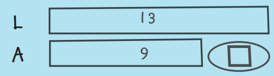
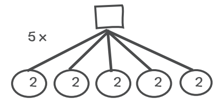
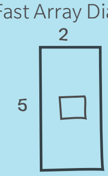
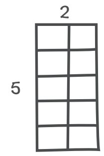
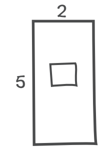
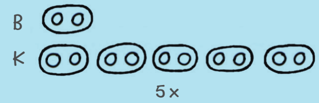
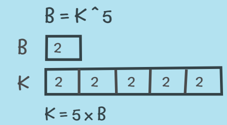
Math Expressions exposes students to many different types of ambitious word problems that are used internationally. The three main types of addition and subtraction situations are shown in Figure 1: Add To/Take From, Put Together/Take Apart, and Comparison problems. The three main types of multiplication and division situations are shown in Figure 2: Equal Groups, Array/Area, and Comparison problems. For all of these types of word problems, each of the three quantities can be the unknown.

Math Expressions encourages teachers to use a variety of activities that will help students understand word problems by using and increasing their linguistic and real-world knowledge. Activities vary according to grade level, but may include acting out the word problem situation, retelling the word problem in their own words, or asking the question using different mathematical terms to practice using all of the important math language. Students also frequently make up word problems.

REPRESENT THE SITUATION

Math Expressions encourages students to use math drawings and visual models and tools to represent the problem situation. In Kindergarten, children use objects and fingers to show problem situations. But from Grade 1 on, students make math drawings to show the problem situation. They use circles or other simple shapes, and students label each part to show how it relates to the situation. Examples of math drawings for the simplest unknown for each problem type are shown in

Figure 3. Representing Addition, Subtraction, Multiplication, and Division Situations

Problem Type	Word Problem	Representation	
		Math Drawing	**Diagram**
Add To	Dan had 9 cherries. Then he picked 4 more. How many does he have now?		9 + 4 = □ (situation/solution equation)
Take From	Dan had 13 cherries. Then he ate 9 of them. How many does he have now?		13 - 9 = □ (situation/solution equation)
Put Together/ Take Apart	Ana has 9 dimes and 4 nickels. How many coins does she have in all?		Math Mountain Diagram 
Additive Comparison	Ali has 9 balloons. Lisa has 13 balloons. How many more balloons does Lisa have than Ali?	Matching Drawing 	Comparison Bars 
Equal Groups	Amy has 5 cousins. She is making 2 puppets for each cousin. How many puppets will Amy need to make?	Grouping Model 	Equal Shares Diagram 
Array	A garden has 5 rows and 2 columns of bean plants. How many plants are there in all?	Array Model 	Fast Array Diagram 
Area	The garden is 5 yards on one side and 2 yards on the side touching this. What is its area?	Area Model 	Fast Area Diagram 
Multiplicative Comparison	Bill has 2 apples. Kim has 5 times as many apples as Bill. How many apples does Kim have?	Grouping Model 	Comparison Bars 

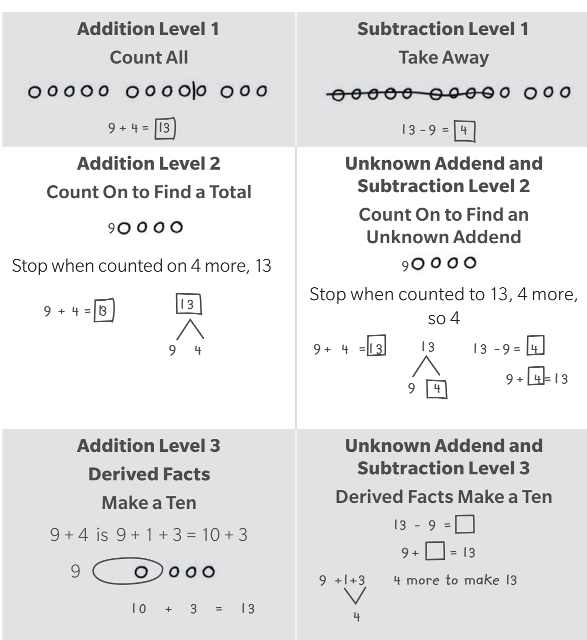
Note: Children find it so natural to show Add To and Take From situations with equations that such situation equations are used as the numerical "diagram" [visual model].

Figure 3. Representing Addition, Subtraction, Multiplication, and Division Situations.

As numbers get larger and children's solution methods become more advanced, situation drawings need to emphasize the relationships among the quantities in the situation. A major focus of the research underlying *Math Expressions* was on powerful diagrams for each problem type. These are shown in the final column of Figure 3. These can be used with whole numbers of any size and with decimals and fractions. They also lend continuity to problem solving throughout the grades. Research indicates that students spontaneously use equations to represent all of the addition and subtraction situations given in Figure 1. These are called situation equations and are shown in Figure 1 for each problem type. Students think of comparison situations in various ways and may use equations not shown in the table.

Easy problems are those in which the situation equation also is the solution equation, such as 9 + 4 = ☐ or 13 − 9 = ☐. For medium and difficult problems, students represent the situation with a situation equation, math drawing, or diagram (or a drawing or diagram and also an equation) and then work from that to find the answer. Sometimes they write a solution equation, but sometimes they may just write the computation.

Figure 4. Levels of Addition and Subtraction Solution Methods



Note: Children can use their fingers to keep track of how many they counted on.

SINGLE-DIGIT ADDITION AND SUBTRACTION

Students around the world move through levels of more advanced solutions for single-digit addition and subtraction (see Figure 4). These levels affect the math drawings made by students and the kinds of problems they can solve. At Level 1, students must represent each of the three quantities separately. They can solve the easy types: Result Unknown and Total Unknown. At Level 2, students can embed one addend inside a total. This enables them to count on to solve easy problems and those with unknown addends. They also can see subtraction as an unknown addend and use counting on to solve subtraction; this is much easier than counting down. Students may use fingers to keep track of how many they counted on. At Level 3, students can relate three addends within a total and therefore can change one problem to a related problem, such as using the general make-a-ten methods shown in Figure 4.

Students also move through levels of solving single-digit multiplication and division problems, but this does not affect problem types. It is helpful for students to understand division as finding an unknown factor, and some students represent division problems with such equations: 5 x ☐ = 10 or ☐ x 5 = 10 instead of 10 ÷ 5 = ☐.

SINGLE-DIGIT MULTIPLICATION AND DIVISION

Extensions of the three addition and subtraction levels to multiplication and division are shown in Figure 5. At Level 1, there are two Equal Groups division methods that directly model the two division situations shown in Figure 2. The first Number of Groups Unknown

Figure 5. Levels of Multiplication and Division Solution Methods

Multiplication

Level 1 Count All

A)	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	3 x 5
B)	(1)	(2)	(3)	
C)	1 2 3 4 5	6 7 8 9 10	11 12 13 14 15	=15

Level 2 Count by *n*

A)	5	10	15	3 x 5
B)	(1)	(2)	(3)	=15

Stop when you count 3 fives.
Answer is the last count by.

Level 3 Recompose (Decompose and Compose)

10 + 5	3 x 5 = (2 + 1)5
(2) + (1)	= 10 + 5 = 15

Division

Level 1 Count All

A)	1 2 3 4 5	6 7 8 9 10	11 12 13 14 15	15 ÷ 5
B)	(1 2 3 4 5)	(1 2 3 4 5)	(1 2 3 4 5)	
C)	(1)	(2)	(3)	=3

Level 2 Count by *n*

A)	5	10	15	15 ÷ 5
B)	(1)	(2)	(3)	=3

Stop when you hear 15.
Answer is the number of groups.

Level 3 Recompose (Decompose and Compose)

10	+	5	15 ÷ 5 = 10 + 5
(2)	+	(1)	= (2) + (1) = 3

Note: *(n)* is counting groups not individual objects. The answer is in red.

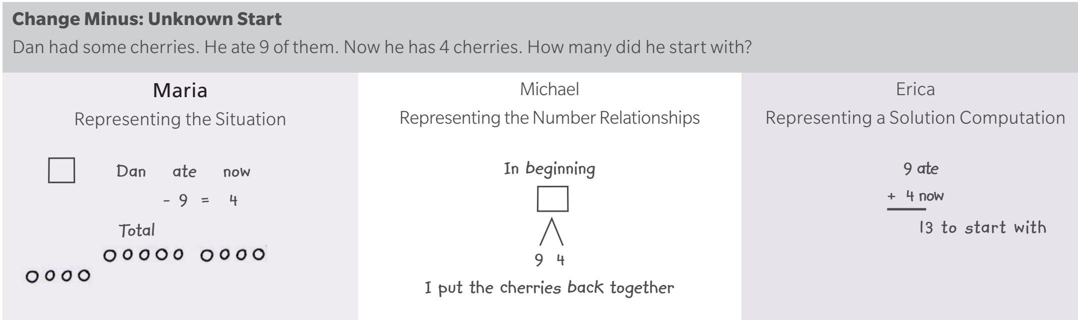
situation is shown in Figure 5 for 15 ÷ 5 = 3. For Level 2 one must count by the known factor even if it is the number of groups. In *Math Expressions* students see equal groups situations as arrays to understand that the roles of the factors can be interchanged; this supports their use of Level 2 methods.

There are many patterns involved in multiplication/division learning; these reflect how the group size relates to the ten in our base-ten numbers (e.g., 5 and 10 patterns are easy, 7 is difficult). Multiplications and divisions can be learned at the same time. At Level 2 a division can be easier than a multiplication, but the reverse is the case at Level 3. Computational resources merge into rich knowledge of multiplicative structure of numbers ≤ 100, and strategies become difficult to differentiate, particularly for smaller factors. *Math Expressions* uses extensive visual, pattern, and situational analysis by students followed by extensive work to bring all students to fluency.

VARIABILITY IN STUDENT PROBLEM SOLVING

Students represent and solve problems in different ways. Figure 6 shows three different solutions of second graders to a difficult Change Minus: Unknown Start problem. *Math Expressions* emphasizes labeling the parts of a drawing or equation to relate it to the parts of the situation (earlier examples in the figures were not labeled because of lack of space). This is a crucial part of algebraic problem solving that prepares students for later use of letters as unknowns and as variables. The labels also facilitate Math Talk about solutions because they capture important aspects of thinking during the problem-solving process. Maria (see Figure 6) represented the situation with her own math drawing and with a situation equation. Michael used the Math Mountain drawing to represent the relationships among the numbers. He could represent the situation in his head and reflect on it to know that he needed to put the addends back together to make the unknown total. Erica was even more advanced and

Figure 6. Variability in Representing and Solving



just wrote a solution computation. Together, these student representations show the whole process of problem solving and enable classmates to make connections and move to more advanced solutions as they listen to Math Talk explanations.

It is important for students to understand where the addends and the total are located in addition and subtraction equations. This enables a student to visualize the situation equation and operate on it to draw a Math Mountain or write a computation.

Students may especially differ in how they solve addition comparison problems. Students may make a matching drawing or draw Comparison Bars, as in Figure 3. They may also make a situation equation, $9 + \square = 13$, or a solution equation, $13 - 9 = \square$ (though the first equation is also a solution equation for students who solve subtraction by finding an unknown addend). Math Talk in a classroom allows students to build relationships among all of the different equations and drawings. This permits students to relate to the math situation in their own way. The drawings facilitate student explanations, help listeners comprehend these explanations, give teachers insight into their students' mathematical thinking, eliminate the cost and logistical issues that arise with manipulatives, and leave students and teachers with a durable record of student thinking.

CHECK SITUATIONS FOR REASONABLENESS

Now that students have understood, represented, and solved the problem, they must determine that their answer is reasonable and justify their mathematical thinking to others. To do this, students use the Math Talk structure (See Math Talk Community in *Math Expressions* for more details on implementing Math Talk in the classroom).

When explaining, discussing, questioning, and justifying their math thinking, students use math drawings, diagrams, equations, and/or solution methods in order to solve the problems. These visual models allow students to effectively explain their thinking and help other students and teachers to understand their thinking. Students solve and explain problem-solving methods in pairs, small groups, or in front of the class at the board or with individual MathBoards. They also respond to questions from the class or teacher and justify their work.

Communication (Math Talk) in teaching problem solving is very beneficial for students. Math Talk helps students to clarify their own thinking and assist other students, provides ongoing formative assessment opportunities for teachers, and allows students to learn from their own errors and errors made by other students.

This program will enable you to advance your students through the learning path levels, thus supporting students from different backgrounds to learn ambitious levels of mathematics with understanding, fluency, and confidence.

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